OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

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CHAPTER-13 : RECTANGULAR CARTESIAN CO-ORDINATE SYSTEMS AND PAIR OF STRAIGHT LINES (FIRST DEGREE)

UNIT TEST-1

1. If the equation of the pair of straight lines passing through the point (1, 1), and making an angle θ with the positive direction of x-axis and the other making the same angle with the positive direction of y-axis is $x^2 - (a+2)xy + y^2 + x(x+y-1) = 0$, $a \; - 2$, then the value of $\sin 2\theta$ is

2. If two of the lines represented by $x^4 + x^3y + cx^2y^2 - xy^3 + y^4 = 0$ bisect the angle between the other two then find the value of c.

3. If x, y, z are different from 1 (one) and are the roots of the equation $t^3 + t^2 + t - 4 = 0$ then the points $\left(\frac{x^3}{x-1}, \frac{x^2-3}{x-1}\right), \left(\frac{y^3}{y-1}, \frac{y^2-3}{y-1}\right), \left(\frac{z^3}{z-1}, \frac{z^2-3}{z-1}\right)$ are collinear.

4. The triangle ABC, right angled at C, has median AD, BE and CF. AD lies along the line $y = x + 3$, BE lies along the line $y = 2x + 4$, If the length of the hypotenuse is 60, find the area of the triangle ABC.

5. Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin are concurrent. Dose this result also hold for the curve, $3x^2 + 3y^2 + 2x + 4y = 0$? If yes, what is the point of concurrency and if not, give reasons.

6. P is the point (– 1, 2), a variable line through P cuts the x and y axes at A and B respectively. Q is the point on AB such that PA, PQ, PB are in HP. Find the locus of Q.

7. The equations of the altitudes AG, BE, CF of a triangle ABC are $x + y = 0$, $x + 4y = 0$ and $2x - y = 0$ respectively. The coordinates of A are $(t, -t)$. Find coordinates of B & C. prove that if t varies the locus of the centroid of the triangle ABC is $x + 5y = 0$.

8. Let the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable straight line be zero. Then the line passes through a fixed point whose co-ordinates are .....

9. If the straight lines $ax + by + p = 0$ and $x \cos\alpha + y \sin\alpha - p = 0$ enclose an angle $\pi/4$ between them and meet the straight line $x \sin\alpha - y\cos\alpha = 0$ in the same point, then the value of $a^2 + b^2$ is equal to .....

10. If $4A^2 + 9B^2 - C^2 + 12AB = 0$, then the family of straight lines $Ax + By + C = 0$ is either concurrent at ... or at ....

11. Prove that the lines joining the origin to the points of intersection of circle $(x - h)^2 + (y - k)^2 = k^2$ with the line $\frac{x}{h} + \frac{y}{k} = 2$ will be perpendicular if (h, k) lies on the circle $x^2 + y^2 = r^2$.

12. Let a and b be non-zero real numbers. Then the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

(a) four straight lines, when $c = 0$ and a, b are of the same sign

(b) two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a

(c) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(d) a circle and an ellipse, when a and b are of the same sign and c is of a sign opposite to that of a

13. Find the equations of the lines joining the origin to the points of intersection of the lines $y = mx + c$ with the circle $x^2 + y^2 = a^2$ and find the condition for these lines to be perpendicular.

14. The portion of the line $lx + my = 1$ intercepted by the circle $x^2 + y^2 = a^2$ subtends an angle of 45° at the centre of the circle. Prove that $4[a^2(l^2 + m^2) - 1] = [a^2(l^2 + m^2) - 2]^2$

Hints and Solutions

1. (a) As both line passes through (1, 1) and one line makes angle θ with x-axis and other line with y-axis slopes of line are tan θ and cot θ

Equations of the given lines are $y - 1 = \tan\theta(x - 1)$ and $y - 1 \cot\theta\,(x - 1)$

So, their combined equation is $[(y - 1) - \tan\theta(x - 1)]\,[(y - 1) - \cot\theta\,(x - 1)] = 0$

$\Rightarrow (y - 1)^2 - (\tan\theta + \cot\theta)(x - 1)(y - 1) + (x - 1)^2 = 0$

$\Rightarrow x^2 - (\tan\theta + \cot\theta)\,xy + y^2 + (\tan\theta + \cot\theta - 2)$

$(x + y - 1) = 0$

Comparing with the given equation we get $\tan\theta + \cot\theta = a + 2$

$$\Rightarrow \frac{1}{\sin\theta\cos\theta} = a + 2 \Rightarrow \sin 2\theta = \frac{2}{a+2}$$

2. (a) As the product of the slopes of the four represented by the given equation is 1 and a pair of line represents the bisectors of the angles between the other two, the product of the slopes of each pair is – 1.

So let the equation of one pair be $ax^2 + 2hxy - ay^2 = 0$.

The equation of its bisectors is $\dfrac{x^2 - y^2}{2a} = \dfrac{xy}{h}$.

By hypothesis $x^4 + x^3y + cx^2y^2 - xy^3 + y^4$

$= (ax^2 + 2hxy - ay^2)(hx^2 - 2axy - hy^2)$

$= ah\,(x^4 + y^4) + 2\,(h^2 - a^2)(x^3y - xy^3) - 6ahx^2y^2$

Comparing the respective coefficients,

we get $ah = 1$

and $c = -6ah = -6$

3. (a) $$\Delta = \begin{vmatrix} \frac{x^3}{x-1} & \frac{x^2-3}{x-1} & 1 \\ \frac{y^3}{y-1} & \frac{y^2-3}{y-1} & 1 \\ \frac{z^3}{z-1} & \frac{z^2-3}{z-1} & 1 \end{vmatrix}$$

Multiplying R_1, R_2, R_3 by $x - 1, y - 1, z - 1$ respectively,

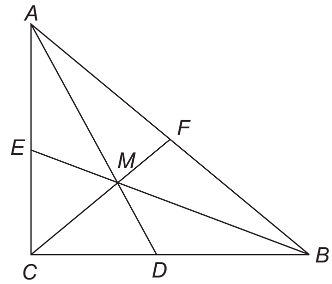
$$\Delta = \begin{vmatrix} x^3 & x^2-3 & x-1 \\ y^3 & y^2-3 & y-1 \\ z^3 & z^2-3 & z-1 \end{vmatrix} (x-1)(y-1)(z-1)$$

Apply $C_1 + C_2 + C_3$ then the new column C_1 becomes of zeros as x, y, z are roots of $t^3 + t^2 + t - 4 = 0$

$\therefore \quad \Delta = 0$

Hence the statement is true.

4. (a) Area (ABC) = 3 Arc (AMB)



In ΔACB

$$AF = BF = CF\ \frac{AB}{2} = 30$$

$$CM : MF = 2 : 1$$

$$\Rightarrow \quad MF = 10$$

By Apollonius theorem

$$AM^2 + BM^2 = 2\,(CF^2 + AF^2)$$

$$AM^2 + BM^2 = 2000$$

Let $< AMB = \theta$

$$\tan\theta = \frac{M_{BE} - M_{AD}}{1 + M_{BE}M_{AD}} = \frac{2-1}{1+2} = \frac{1}{3}$$

$$\Rightarrow \cos\theta\frac{3}{\sqrt{10}} \text{ and } \sin\theta = \frac{1}{\sqrt{10}}$$

$$\cos\theta = \frac{AM^2 + BM^2 - AB^2}{2AM.BM}$$

$$\frac{3}{\sqrt{10}} = \frac{2000 - 3600}{2AM.BM}$$

$$\Rightarrow AM.BM = -\frac{1600}{6}\sqrt{10}$$

Area of $\Delta AMB = \dfrac{1}{2}(AM)(BM)\sin\theta$

$$= \frac{1}{2} \times \frac{1600}{6}\sqrt{10} \times \frac{1}{\sqrt{10}} = \frac{400}{3}$$

Area of ABC 3 × Area of ΔAMB

$$= 3 \times \frac{400}{3} = 400 \text{ Sq. units}$$

5. (a) 15 : $y = mx + c$

$$3x^2 - y^2 - 2x\left(\frac{y - mx}{c}\right) + 4y\left(\frac{y - mx}{c}\right) = 0$$

$$3x^2 - y^2 - \frac{2xy}{c} + \frac{2mx^2}{c} + \frac{4y^2}{c} - \frac{4mxy}{c} = 0$$

$$x^2\left(3 + \frac{2m}{c}\right) + y^2\left(\frac{4}{c} - 1\right) + xy\left(-\frac{4m}{c} - \frac{2}{c}\right) = 0$$

Now $\frac{a}{b} = -1$

$\Rightarrow \quad \frac{3c+2m}{4-c} = -1$

$\Rightarrow \quad 3c + 2m = c - 4$

$\Rightarrow \quad 2c + 2m = -4$

$\Rightarrow \quad c + m = -2$

Point is (1, – 2)

For $3x^2 + 3y^2 + 2x + 4y = 0$

Equation will be

$$x^2\left(3 - \frac{3m}{c}\right) + y^2\left(\frac{4}{c} - 1\right) + xy\left(\frac{-4m}{c} + \frac{2}{c}\right) = 0$$

$$\frac{3c-2m}{4-c} = -1$$

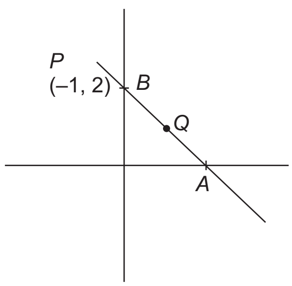
$\Rightarrow \quad 3c - 2m = c - 4$

$c - m = 2$

$\Rightarrow$ Point is (– 1, – 2)

6. (a) The equation of line $(y - 2) = m(x + 1)$

$\Rightarrow \quad A = \left(\frac{-2}{m} - 1, 0\right); B = (0, 2+m)$



Assume $Q = (h, k)$

Q is on the line

$\Rightarrow \quad k = 2m(h + 1)$

$$PQ = \sqrt{(h+1)^2 + (k-2)^2}$$

$$= \sqrt{(h+1)^2 + m^2(h+1)^2} = |h+1|\sqrt{m^2+1}$$

$$PA = \sqrt{\left(\frac{-2}{m} - 1 + 1\right)^2 + 2^2} = \left|\frac{2}{m}\right|\sqrt{1+m^2}$$

$$PB = \sqrt{1+m^2}$$

It's given that PA, PQ, PB are in H. P.

$$\frac{2}{PQ} = \frac{1}{PA} = \frac{1}{PB}$$

$$\Rightarrow \frac{2}{|h+1|\sqrt{m^2+1}} = \left|\frac{m}{2}\right|\frac{1}{\sqrt{1+m^2}} + \frac{1}{\sqrt{1+m^2}}$$

$$\Rightarrow \frac{2}{|h+1|} = \frac{1}{2}\left|\frac{k-2}{h+1}\right| + 1$$

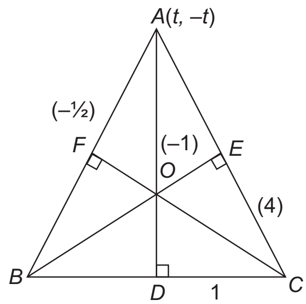
$\Rightarrow 2|x+1| + |y-2| = 4$

If $m < 0$ then $= 2x$

If $m > 0$ then $2x + y = 4$ (If $x > 1$ and $y > 2$)$= 4$

(If $x < 1$ and $y < 2$)

7. (a)



$m_{BC} = 1$

$m_{AC} = 4 \Rightarrow y = 4x - 5t$

$m_{AB} = \frac{-1}{2}$

$\Rightarrow \quad 2y + x + t = 0$

Solving AB and BE we get $x = -4y$

$$t = 2y \Rightarrow y = \frac{t}{2} \Rightarrow B = \left(\frac{-4t}{2}, \frac{t}{2}\right)$$

CF and $AC \Rightarrow y = 2x$

$$2x = 5t \Rightarrow C = \left[\frac{5t}{2}, 5t\right]$$

$G = (h, k)$

$$3h = t + \frac{5t}{2} - \frac{4t}{2}; 3k = t + 5t + \frac{t}{2}$$

$$h = \frac{t}{2} = \frac{3t}{2} \Rightarrow k = 3h$$

8. (a) $p_1 + p_2 + p_3 = 0$

$\Rightarrow 3(a + b + c) = 0$

$\therefore$ Line $ax + by + c = 0$ passes through (1,1).

9. (a) $\tan\frac{\pi}{4} = \frac{a\sin\alpha - b\cos\alpha}{a\cos\alpha + b\sin\alpha} = 1$

$\therefore a\sin\alpha - b\cos\alpha = a\cos\alpha + b\sin\alpha$ ...(1)

The condition of concurrency *i.e.,* $\Delta = 0$ implies

$a\cos\alpha + b\sin\alpha = 1$ ...(2)

Hence from (1)

$a\sin\alpha - b\cos\alpha = 1$ [by (2)]

Squaring and adding (2) and (3), $a^2 + b^2 = 2$

10. (a) (2,3), (– 2, – 3).

The given relation can be written as

$$(2A + 3B)^2 - C^2 = 0$$

or $(2A + 3B + C)(2A + 3B - C) = 0$

$\therefore \quad 2A + 3B + C = 0$

or $\quad -2A - 3B + C = 0$

Above relation $\Rightarrow$ that the family of lines

$$Ax + By + C = 0$$

concur at the point (2, 3) or (– 2, – 3).

11. (a) Making the equation of circle homogeneous.

$$x^2 + y^2 - 2(hx + ky), \frac{1}{2}\left(\frac{x}{h} + \frac{y}{k}\right)$$

$$= \left(h^2 + k^2 - r^2\right) \cdot \frac{1}{4}\left(\frac{x}{h} + \frac{y}{k}\right)^2$$

$$A + B = 0 \Rightarrow \left(\frac{h^2 + k^2 - r^2}{4}\right)\left(\frac{1}{h^2} + \frac{1}{k^2}\right) = 0$$

$$\therefore \quad h^2 + k^2 - r^2 = 0, as \frac{1}{h^2} + \frac{1}{k^2} \neq 0$$

$\therefore$ Locus of (h, k) is $x^2 + y^2 = r^2$.

12. (b) $x^2 - 5xy + 6y^2 = 0$

$\Rightarrow \quad (x - 3y)(x - 2y) = 0$

Above represents two straight lines through (0,0).

$$ax^2 + by^2 + c = 0$$

$$\Rightarrow \quad x^2 + y^2 = \frac{c}{a}$$

where, $c = 0$ and a and b are of same sign *i.e.*, of the form $5x^2 + 9y^2 = 0$. This not possible.

When $a = b$ and c is of sign opposite to that of a then from (1), 2nd factor is $x^2 + y^2 = +$ ive $= k^2$. Hence a circle $\Rightarrow$ (b) is correct.

Both (c)and (d) are obviously incorrect.

13. (a) From the equation $\frac{y - mx}{c} = 1$.

$\therefore x^2 + y^2 = a^2 . l^2,$

$$\text{or } x^2 + y^2 = a^2 \frac{(y - mx)^2}{c^2}$$

$(c^2 - a^2m^2)x^2 + 2am^2 xy + (c^2 - a^2)y^2 = 0.$

If it represents two perpendicular straight lines, then coeff. of x^2 + coeff. of $y^2 = 0$.

$$c^2 - a^2 m^2 + c^2 - a^2 = 0$$

$2c^2 = a^2 (1 + m^2)$ is the required condition.

14. The centre of the circle is origin. The equations of the lines joining origin to the points of intersection of the line and circle are

$$x^2 + y^2 = a^2 (lx + my)^2$$

$$\text{or } x^2 (a^2 l^2 - l) + 2\, lm\, a^2 xy + y^2 (a^2 m^2 - l) = 0$$

$$Ax^2 + 2Hxy + By^2 = 0$$

$$\tan 45° = 1 = \frac{2\sqrt{H^2 - AB}}{4 + B}$$

$$(A + B)^2 = 4\,[H^2 - AB]$$

$$[a^2 (l^2 + m^2) - 2]^2 = 4\,[a^2 (l^2 + m^2) - 1]$$